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Computation of Efficient Compromise Arcs in Convex Quadratic Multicriteria Optimization*

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Abstract. Given a Convex Quadratic Multicriteria Optimization Problem, we show the stability of the Domination Problem. By modifying Benson's single parametric method, which is based on the Domination Problem, we are able to show the existence of an efficient compromise arc connecting any two efficient points. Moreover, we deduce an algorithm which realizes the modification in polynomial time.

Key words: Benson's method, Domination problem, Multicriteria optimization, Quadratic functions, Stability

1. Introduction

In this paper, we study continuity properties of the Multicriteria Optimization Problem

$$\mathrm{MP} : \max !\{f(x) | x \in X\}.$$

Our investigation will be focused on the Convex Quadratic Multicriteria Optimization Problem (QMP), i.e., $X \subset \mathbb{R}^n$ is a convex polyhedron and the concave quadratic functions $f = (f_1, \dots, f_q)^{\mathrm{T}} : \mathbb{R}^n \to \mathbb{R}^q$ are to be maximized on X. This problem class frequently arises in Stochastic Multicriteria Optimization [4] and its various applications to Risk Management [18].

If no further information about the preferences of the decision maker (DM) is given, then the solutions to the MP usually considered are efficient points. However, sometimes the DM requires the solution in addition to dominate a "least" outcome value $v \in f(X) - \mathbb{R}^q_+$

$$\mathrm{MP}(v) \colon \max !\{f(x) | x \in X, f(x) \geqslant v\},$$

i.e., f is to be maximized on $X(v)$, where

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$$X(v) := \{x \in X | f(x) \geqslant v\}.$$

This parametric MP will be referred to as the *Domination Problem*, and we will consider the efficient set as it's global solution set. Again, if the target functions are concave quadratic and the constraint set is a convex polyhedron, the Domination Problem will be denoted by QMP(v).

In this context, an important property is the *Domination Property*, that is, the Domination Problem is solvable for any $v \in f(X) - \mathbb{R}^q_+$. Helbig [12] has shown that for the QMP, $f(X) - \mathbb{R}^q_+$ is closed convex, and Henig [13] has shown that under this precondition, the Domination Property holds if the set of efficient points is not empty. In this discussion, we will generally assume the efficient set to be nonempty, thus yielding the Domination Property for QMP(v).

Another important property is the stability of the Domination Problem, i.e., the continuity of its efficient set mapping. This property is particularly desirable, for minor errors of the least outcome value (errors in preference estimation or numerical computing) should only result in minor variation of the efficient set.

Based on the Domination Problem, interactive algorithms for decision support in Multicriteria Optimization have been developed. Benson's method [3, 8] bears the distinct advantage of computing only efficient points (contrary to weakly efficient points, cf.[5]). In each step $k \in \mathbb{N}$, the DM fixes a least outcome value $v^k \in f(X) - \mathbb{R}^q_+$ and a solution x^k to the weighted sum scalarized Domination Problem is computed. The procedure is repeated until the DM accepts x^k as his preferred solution to the MP. Guddat and Guerra [10] analyzed the method for single parametric variation of the outcome value, i.e., parametrically solving the weighted sum scalarized Domination Problem on the line $[v^k, v^{k+1}] \subset f(X) - \mathbb{R}^q_+$, for this generates additional information about efficient compromises between v^k and v^{k+1}. This information can be employed to aid the DM in selecting a solution when he/she is unsure about his/her preferences, or to find compromises between the extreme positions of two DMs. We will refer to this procedure as Benson's single parametric method.

Concerning this parametric procedure, the existence of a continuous selection arc for the efficient set mapping is of special importance, for in this case the DM is able to continuously adjust his preferred solution by adjusting the outcome value. This selection arc will be called efficient compromise arc. In the general MP, Guddat and Guerra [10] have investigated several conditions for the existence of such an arc.

We will first show the stability of the Domination Problem and of the weighted sum scalarized Domination Problem for the QMP. By applying this result to the single parametric Domination Problem, we are able to deduce the existence of an efficient compromise arc connecting any two efficient points without the conditions imposed in [10].

Moreover, we show that by minor modification, Benson's single parametric method can be employed to compute efficient compromise arcs, and that the modification's numerical realization only requires polynomial time.

2. Preliminaries

2.1. NOTATION

Throughout this discussion, let $\mathbb{R}^q$ be associated with the Euclidean norm $\|\cdot\|$. $\mathbb{R}^q_+$ stands for the nonnegative orthant in $\mathbb{R}^q$. We denote the (open) Euclidean ball in $\mathbb{R}^q$ of radius $\varepsilon > 0$, centered at $z \in \mathbb{R}^q$, by $B_\varepsilon(z)$. int M and cl M stand for the interior and closure of a set $M \subset \mathbb{R}^q$, respectively. For $A, B \in \mathbb{R}^n$, let $A + B := \{a + b | a \in A, b \in B\}$. Notions of convex analysis (e.g., convex, concave, polyhedron) and elementary results will be employed without reference, standard texts (cf. Rockafellar's excellent monograph [19]) discuss those results that we use. Here and in what follows, denote the kernel of a $m \times n$ matrix $A \in \mathbb{R}^{m\times n}$ by ker A, the transposition by A^{T}, and the level set mapping of a function $f : X \to \mathbb{R}^q$ by $f^{-1}(z) := \{x \in X | f(x) = z\}, z \in \mathbb{R}^q$. If an assumption holds without loss of generality, we will use the abbreviation "w.l.o.g.".

DEFINITION 2.1. Let $V \subset \mathbb{R}^q$ and $\psi : V \to 2^{\mathbb{R}^n} \backslash \emptyset$ be a set-valued mapping. ψ is called

(C1) upper semicontinuous iff for each $v \in V$ and for each neighborhood W of $\psi(v)$ there is a neighborhood U of v such that $\psi(V \cap U) \subset W$.
(C2) lower semicontinuous iff for each $v \in V$, $w \in \psi(v)$ and for each neighborhood W of w there is a neighborhood U of v such that $\psi(u) \cap W \neq \emptyset$ for all $u \in U \cap V$.
(C3) continuous iff ψ is upper and lower semicontinuous.

2.2. MULTICRITERIA OPTIMIZATION

For z^0, $z^1 \in \mathbb{R}^q$, define $z^1 \geqslant z^0$ (resp. $z^1 > z^0$) by componentwise "$\geqslant$" (resp. "$>$"), define $z^1 \ngeq z^0$ by $z^1 \geqslant z^0$ and $z^1 \neq z^0$. We will employ the following efficiency concepts:

(E1) Efficient points $x^0 \in X$: There is no $x^1 \in X$ such that $f(x^1) \ngeq f(x^0)$.
(E2) Properly efficient points $x^0 \in X$: There is $M > 0$ such that for each $i \in \{1, \ldots, q\}$ and each $x^1 \in X$ with $f_i(x^1) > f_i(x^0)$ there is $j \in \{1, \ldots, q\}, j \neq i$, satisfying

$$f_j(x^1) < f_j(x^0) \quad \text{and} \quad (f_i(x^1) - f_i(x^0))/(f_j(x^0) - f_j(x^1)) \leqslant M$$

(E3) Weakly efficient points $x^0 \in X$: There is no $x^1 \in X$ such that $f(x^1) > f(x^0)$.

The set of efficient (respectively properly efficient, weakly efficient) points x^0 is denoted by $E_f(X)$ (resp. $P_f(X)$, $W_f(X)$), the set of efficient (resp. properly efficient, weakly efficient) outcomes $f(x^0)$ is denoted by $E(f(X))$ (respectively $P(f(X))$, $W(f(X))$). The following theorem is due to Arrow, Barankin and Blackwell [1].

THEOREM 2.1. *Let* $X \subset \mathbb{R}^n$ *be convex*, $f : X \to \mathbb{R}^q$ *be (componentwise) concave. Then,*

$$P(f(X)) \subset E(f(X)) \subset \operatorname{cl} P(f(X)).$$

The optimal sets X_λ of the weighted sum scalarized problem

$$\mathrm{MP}_\lambda: \ \max!\{\lambda^{\mathrm{T}} f(x) | x \in X\}, \quad \lambda \in \mathbb{R}^q_+ \setminus \{0\}, \tag{1}$$

are usually considered to characterize properly and weakly efficient points (cf. [9], [15]).

THEOREM 2.2. *Let* $X \subset \mathbb{R}^n$ *be convex*, $f{:}\mathbb{R}^n \to \mathbb{R}^q$ *be concave. Then,*

$$P_f(X) = \bigcup_{\lambda \in \operatorname{int}\mathbb{R}^q_+} X_\lambda, \quad W_f(X) = \bigcup_{\lambda \in \mathbb{R}^q_+ \setminus \{0\}} X_\lambda.$$

In the following, set $Z := f(X)$ and $Z(v) := (Z - \mathbb{R}^q_+) \cap (v + \mathbb{R}^q_+)$, $v \in Z - \mathbb{R}^q_+$. Then in general, $Z(v) \neq f(X(v))$, but the efficient points coincide (cf. [8]).

Choose $\lambda \in \operatorname{int}\mathbb{R}^q_+$. Then by $X_\lambda(v)$, respectively, $Z_\lambda(v) := f(X_\lambda(v))$, denote the optimal set, respectively the optimal outcome set, of the weighted sum scalarized Domination Problem

$$\mathrm{MP}_\lambda(v){:}\ \max!\{\lambda^{\mathrm{T}} f(x) | x \in X(v)\}, \quad v \in Z - \mathbb{R}^q_+. \tag{2}$$

Since for the QMP, $Z - \mathbb{R}^q_+$ is closed convex and we assume $E(Z)$ to be nonempty, it is well known (cf. [13]) that $Z(v)$ is compact, hence the sets $Z_\lambda(v)$ and $X_\lambda(v)$ are nonempty for each $v \in Z - \mathbb{R}^q_+$. In the course of our investigation, the target function $f : \mathbb{R}^n \to \mathbb{R}^q$ of the QMP will be generally assumed to be given by

$$f_i(x) = x^{\mathrm{T}} C_i x + c^{i\mathrm{T}} x + \gamma_i, \quad x \in X,$$

where $C_i \in \mathbb{R}^{n \times n}$ is negative semidefinite, $c^i \in \mathbb{R}^n$, $\gamma_i \in \mathbb{R}$, $i = 1, \ldots, q$. We will require the following theorems (cf. [14]).

THEOREM 2.3. *Let $X \subset \mathbb{R}^n$ be a convex polyhedron, $r \in \mathbb{R}^n \setminus \{0\}$. Then, the mapping $\pi : X \to \mathbb{R} \cup \{\infty\}$ defined by $\pi(x) := \sup\{t \in \mathbb{R} | x + tr \in X\}$ satisfies exactly one of the following assertions*:

(i) $\pi(x) = \infty \quad \forall x \in X$.
(ii) $\pi(x) < \infty \quad \forall x \in X$ *and π is continuous.*

THEOREM 2.4. *Let $X \subset \mathbb{R}^n$ be a convex polyhedron, $f : \mathbb{R}^n \to \mathbb{R}^q$ be concave quadratic. Moreover, assume that $(z^k)_{k \in \mathbb{N}} \subset E(f(X))$ converges to $z \in E(f(X))$. Then, for each $x \in f^{-1}(z)$ there is a selection sequence $x^k \in f^{-1}(z^k)$, $k \in \mathbb{N}$, converging to x.*

3. Stability of the Domination Problem

In this section, we will discuss continuity properties of the efficient outcome set mapping $E(Z(v))$ and the optimal outcome set mapping $Z_\lambda(v)$ for the QMP.

DEFINITION 3.1. The Domination Problem is called *stable* iff the mapping

$$v \mapsto E(Z(v)) = E(Z) \cap (v + \mathbb{R}^q_+), \quad v \in Z - \mathbb{R}^q_+,$$

is continuous.

Obviously, for each $v \in Z - \mathbb{R}^q_+$, $\mathrm{MP}_\lambda(v)$ constitutes a quadratic optimization problem with (concave) quadratic target function and (concave) quadratic constraints.

PROPOSITION 3.1. Let $C_i \in \mathbb{R}^{n \times n}$ be negative semidefinite, $i = 1, \ldots, q$, and $\lambda \in \operatorname{int} \mathbb{R}^q_+$. Then, $\sum_{i=1}^q \lambda_i C_i$ is negative semidefinite, and

$$\ker\left(\sum_{i=1}^{q} \lambda_i C_i\right) \subset \bigcap_{i=1}^{q} \ker C_i.$$

Proof. Obviously, $x^{\mathrm{T}}(\sum_{i=1}^q \lambda_i C_i)x \leqslant 0$ holds for $x \in \mathbb{R}^n$. Moreover, for $x \in \ker(\sum_{i=1}^q \lambda_i C_i)$,

$$\sum_{i=1}^{q} \underbrace{\lambda_i}_{>0} \underbrace{x^{\mathrm{T}} C_i x}_{\leqslant 0} = x^{\mathrm{T}} \left(\sum_{i=1}^{q} \lambda_i C_i\right) x = 0.$$

This implies $x^{\mathrm{T}}C_ix = 0$, and by negative semidefiniteness, $C_ix = 0$, $i = 1, \dots, q$. □

LEMMA 3.1. Let $X \subset \mathbb{R}^n$ be a convex polyhedron, $f: \mathbb{R}^n \to \mathbb{R}^q$ be concave quadratic, $\lambda \in \operatorname{int} \mathbb{R}^q_+$ and $v \in Z - \mathbb{R}^q_+$. Then, for all x^0, $x^1 \in X_\lambda(v)$, $r := x^1 - x^0$,

(i) $C_ir = 0$, $i = 1, \dots, q$,
(ii) $(\sum_{i=1}^q \lambda_i c^i)^{\mathrm{T}} r = 0$.

Proof. Suppose, x^0, $x^1 \in X(v)$ are optimal for the target function

$$Q(y) := y^{\mathrm{T}}\left(\sum_{i=1}^q \lambda_i C_i\right)y + \left(\sum_{i=1}^q \lambda_i c^i\right)^{\mathrm{T}} y + \sum_{i=1}^q \lambda_i \gamma_i, \quad y \in X(v).$$

Then, concavity of Q implies

$$Q(x^0) \leqslant Q(x^0 + tr), \quad t \in [0, 1],$$

and optimality of x^0 implies "$=$". Hence,

$$Q(x^0) = Q(x^0) + t\left(2x^{0\mathrm{T}}\left(\sum_{i=1}^q \lambda_i C_i\right)r + \left(\sum_{i=1}^q \lambda_i c^i\right)^{\mathrm{T}} r\right) + t^2 r^{\mathrm{T}}\left(\sum_{i=1}^q \lambda_i C_i\right)r, \ t \in [0, 1],$$

and by the Fundamental Theorem of Algebra,

(i) $r^{\mathrm{T}}(\sum_{i=1}^q \lambda_i C_i)r = 0$,
(ii) $2x^{0\mathrm{T}}(\sum_{i=1}^q \lambda_i C_i)r + (\sum_{i=1}^q \lambda_i c^i)^{\mathrm{T}} r = 0$.

Now by negative semidefiniteness of $\sum_{i=1}^q \lambda_i C_i$, (i) implies $r \in \ker \sum_{i=1}^q \lambda_i C_i$, and moreover, (ii) implies $r \in \ker(\sum_{i=1}^q \lambda_i c^i)^{\mathrm{T}}$. Thus, the lemma is a consequence of proposition 3.1. □

THEOREM 3.1 *Let $X \subset \mathbb{R}^n$ be a convex polyhedron, $f: \mathbb{R}^n \to \mathbb{R}^q$ be concave quadratic and $\lambda \in \operatorname{int} \mathbb{R}^q_+$. Then, the mapping $v \mapsto Z_\lambda(v)$, $v \in Z - \mathbb{R}^q_+$, is continuous.*

Proof. Since $Z - \mathbb{R}^q_+$ is closed convex, it is easily seen that $Z(v)$, $v \in \mathbb{R}^q_+$, is upper semicontinuous. Since in addition, $Z(v)$ is compact convex and $\lambda^{\mathrm{T}} z$,

$z \in Z(v)$, is linear for all $v \in Z - \mathbb{R}^q_+$, upper semicontinuity of $Z_\lambda(v)$ is a consequence of elementary parametric optimization theory (cf. [2]). We will show that the mapping is also lower semicontinuous, i.e., for all $v \in Z - \mathbb{R}^q_+$, each $w \in Z_\lambda(v)$ is a limit point of a selection sequence in $(Z_\lambda(v^k))_{k\in\mathbb{N}}$, where $(v^k)_{k\in\mathbb{N}} \subset Z - \mathbb{R}^q_+$ converges to v.

Choose $v \in Z - \mathbb{R}^q_+$, $w \in Z_\lambda(v)$ and $(v^k)_{k\in\mathbb{N}} \subset Z - \mathbb{R}^q_+$ converging to v. Moreover, select $z^k \in Z_\lambda(v^k)$, $k \in \mathbb{N}$. Since $Z_\lambda(v)$ is compact and the mapping Z_λ is upper semicontinuous, the sequence z^k converges w.l.o.g. to some $z \in Z_\lambda(v)$. By Theorem 2.4, $z \in E(Z)$ implies that there is a sequence $(x^k)_{k\in\mathbb{N}} \in X$ satisfying $x^k \in f^{-1}(z^k)$, $k \in \mathbb{N}$, which converges to some $x \in f^{-1}(z)$. Choose $y \in f^{-1}(w)$ and set $r := y - x$. Now $\pi : X \to X$, defined by

$$\pi(u) := u + \sup\{t \in [0,1] | u + tr \in X\} \cdot r$$

is continuous by Theorem 2.3. Therefore, the "projected sequence" $y^k := x^k + \alpha_k r : = \pi(x^k)$, $k \in \mathbb{N}$, converges to $y = \pi(x)$, and consequently, $w^k := f(y^k)$ converges to $w = f(y)$. Moreover, by Lemma 3.1,

$$\begin{aligned} w^k = f(x^k + \alpha_k r) &= \begin{pmatrix} {x^k}^T C_1 x^k + {c^1}^T x^k + \alpha_k {c^1}^T r + \gamma_1 \\ \vdots \\ {x^k}^T C_q x^k + {c^q}^T x^k + \alpha_k {c^q}^T r + \gamma_q \end{pmatrix} \\ &= f(x^k) + \alpha_k \begin{pmatrix} {c^1}^T r \\ \vdots \\ {c^q}^T r \end{pmatrix} = z^k + \alpha_k \begin{pmatrix} {c^1}^T r \\ \vdots \\ {c^q}^T r \end{pmatrix}, \quad k \in \mathbb{N}, \end{aligned}$$

and $\sum_{i=1}^q \lambda_i {c^i}^T r = 0$. For

$$H_k := \{u \in \mathbb{R}^q | \lambda^T u = \lambda^T z^k\},$$

this implies $w^k \in H_k \cap Z$. In addition, $Z_\lambda(v^k)$ is obviously given by

$$Z_\lambda(v^k) = H_k \cap Z(v^k) = H_k \cap Z \cap (v^k + \mathbb{R}^q_+).$$

Now

$$w^k - v^k \overset{k\to\infty}{\to} w - v \in \mathbb{R}^q_+,$$

i.e., $d(w^k, v^k + \mathbb{R}^q_+) \to 0$ for $k \to \infty$, and it follows that $d(w^k, Z_\lambda(v^k)) \to 0$ for $k \to \infty$. Since w^k converges to w, there is a sequence $u^k \in Z_\lambda(v^k)$,

$k \in \mathbb{N}$, which is also converging to w. Thus, we have shown that $Z_\lambda(v)$ is lower semicontinuous. □

COROLLARY 3.1 *Let* $X \subset \mathbb{R}^n$ *be a convex polyhedron*, $f : \mathbb{R}^n \to \mathbb{R}^q$ *be concave quadratic. Then, the Domination Problem is stable.*

Proof. Since $E(Z)$ is closed, it is easily seen that $E(Z(v)) = E(Z) \cap (v + \mathbb{R}^q_+)$ is upper semicontinuous. We will again show that the mapping is also lower semicontinuous.

Choose $v \in Z - \mathbb{R}^q_+$, $w \in E(Z(v))$ and $(v^k)_{k\in\mathbb{N}} \subset Z - \mathbb{R}^q_+$ converging to v. By Arrow, Barankin and Blackwell's Theorem 2.1, for each $\varepsilon > 0$ there is $y \in P(Z(v))$ such that $\|w - y\| < \varepsilon/2$. Since y is properly efficient, there is $\lambda \in \text{int}\mathbb{R}^q_+$ satisfying $y \in Z_\lambda(v)$. Then by Theorem 3.1, for k sufficiently large there is $z \in Z_\lambda(v^k) \subset E(Z(v^k))$ satisfying $\|z - y\| < \varepsilon/2$, hence $\|w - z\| < \varepsilon$. This implies lower semicontinuity of $E(Z(v))$. □

4. Application to Benson's Method

Contrary to classical *compromise programming*, where compromises between a feasible point and an ideal or utopia point are considered (cf. [20]), we define any $v \in Z$ to constitute a *t-compromise* between two feasible points v^0, $v^1 \in Z$ iff $v \geqslant v(t)$, where

$$v(t) := (1 - t)v^0 + tv^1, \quad t \in [0, 1].$$

A compromise is called *efficient t-compromise* iff in addition, $v \in E(Z)$. Now Benson's single parametric method can be stated for the QMP as follows (cf. Guddat and Guerra [8]). Recall that for the QMP with $E(Z) \neq$, $Z_\lambda(v) \neq \emptyset$ for all $v \in Z - \mathbb{R}^q_+$.

ALGORITHM 4.1. Let $X \subset \mathbb{R}^n$ be a convex polyhedron, $f: \mathbb{R}^n \to \mathbb{R}^q$ be concave quadratic.

Step 0:

(i) Compute $x \in X$ by applying a method of linear optimization (cf. [7]). If no feasible point can be found, $X = \emptyset$, STOP.
(ii) Choose $\lambda \in \text{int } \mathbb{R}^q_+$ and compute $v^0 \in E(Z)$ by applying a method of quadratic optimization with quadratic constraints (cf. [6,11]) to $\text{MP}_\lambda(f(x))$. If no point in $Z_\lambda(f(x))$ can be found, $E(Z) = \emptyset$, STOP.

Step k, $k \in \mathbb{N}$:

(i) The DM enters the least outcome value $v^k \in \mathbb{R}^q$.
(ii) Define $v(t) := (1 - t)v^{k-1} + \alpha v^k$, $t \in [0, 1]$, and

$t_0 := \max\{t \in [0,1] | v(t) \in Z - \mathbb{R}^q_+\}$.

Compute an *efficient compromise arc* between $v(0)$ and $v(t_0)$, i.e., an arc $\alpha\colon [0, t_0] \to Z$ such that $\alpha(t)$ is an efficient t-compromise for all $t \in [0, t_0]$. The DM chooses his/her preferred efficient compromise $\alpha(t)$.

(iii) If $\alpha(t)$ is accepted by the DM, STOP, else redefine $v^k := \alpha(t)$ and continue with step $k+1$.

The algorithm can be applied by a single DM who would like to explore which efficient solutions are possible depending on the least outcome value. For two DMs, the algorithm could be operated by DM A if k is uneven and by DM B if k is even. Then in each step, an arc of efficient compromises between their preferences is computed.

For ease of notation, we will restrict ourselves to considering step 1 and assuming w.l.o.g. $v^1 \in Z$, else replace v^1 by $v(t_0)$. Then in phase (*ii*), the algorithm solves the single-parametric optimization problem $\mathrm{MP}_\lambda(v(t))$, $t \in [0,1]$. For fixed $N \in \mathbb{N}$, this parametric problem is usually solved by computing solutions to the discrete problem $\mathrm{MP}_\lambda(v(i/N))$, $i = 0, \ldots, N$ (cf. [10]). Obviously each $z \in Z_\lambda(v(t))$, $t \in [0,1]$, is an efficient t-compromise, but in general, arbitrary selection does not yield an efficient compromise arc. Guddat and Guerra [10] analyzed the parametric problem $\mathrm{MP}_\lambda(v(t))$ for the general MP. In their paper, conditions for the existence of a unique solution for each $t \in [0,1]$ are investigated, since in this case the mapping $Z_\lambda(v(t))$, $t \in [0,1]$, constitutes an efficient compromise arc.

In the following, we will again consider the QMP, and construct an efficient compromise arc without the conditions imposed in [10] by employing the lexicographic optimization problem

$$\overline{\mathrm{MP}}_\lambda(v(t))\colon \min!\{\|z\| | z \in Z_\lambda(v(t))\}, \quad t \in [0,1]. \tag{3}$$

Since $Z_\lambda(v(t))$ is compact convex and the Euclidean norm $\|\cdot\|$ is strictly convex, for each $t \in [0,1]$ there is a unique solution $z(t)$ to $\overline{\mathrm{MP}}_\lambda(v(t))$.

THEOREM 4.1. *Let $X \subset \mathbb{R}^n$ be a convex polyhedron, $f\colon \mathbb{R}^n \to \mathbb{R}^q$ be concave quadratic and $\lambda \in \operatorname{int} \mathbb{R}^q_+$. Then*:

(i) $\mathrm{MP}_\lambda(v(t))$ *is stable*, *i.e.*, *the mapping* $t \mapsto Z_\lambda(v(t))$, $t \in [0,1]$, *is continuous*.

(ii) $\overline{\mathrm{MP}}_\lambda(v(t))$ *is stable*, *i.e.*, *the mapping* $t \mapsto z(t)$, $t \in [0,1]$, *is continuous. In particular*, $z(t)$, $t \in [0,1]$, *is an efficient compromise arc*.

Proof. (i) The assertion is a direct consequence of Theorem 3.1.

(ii) Let $(t_k)_{k\in\mathbb{N}} \subset [0,1]$ converge to $t \in [0,1]$. Since $Z_\lambda(v(t))$ is compact, there is an accumulation point $y \in Z_\lambda(v(t))$ of $z(t_k)$. Moreover, by (i) there is a sequence $y^k \in Z_\lambda(v(t_k))$, $k \in \mathbb{N}$, converging to y. Now, in case of $\|y\| < \|z(t)\|$, it follows that $\|y_k\| < \|z(t_k)\|$ for some $k \in \mathbb{N}$, contradicting the definition of $z(t_k)$. Therefore, $\|y\| = \|z(t)\|$ holds, and since $z(t)$ is the unique solution to $\overline{\mathrm{MP}}_\lambda(v(t))$, it follows that $y = z(t)$. □

5. Numerical Realization of the Method

In this section, we will deduce a polynomial time method for the computation of $z(t)$. For each $v \in Z - \mathbb{R}^q_+$, $\mathrm{MP}_\lambda(v)$ constitutes an optimization problem with (concave) quadratic target function and (concave) quadratic constraints, thus the set $X_\lambda(v)$ is given by a cut of finitely many quadrics. We will first show that the sets $X_\lambda(v)$ and $Z_\lambda(v)$ are even convex polyhedra.

THEOREM 5.1. *Let $X \subset \mathbb{R}^n$ be a convex polyhedron, $f: \mathbb{R}^n \to \mathbb{R}^q$ be concave quadratic and $\lambda \in \mathrm{int}\ \mathbb{R}^q_+$. Then for each $v \in Z - \mathbb{R}^q_+$, the sets $X_\lambda(v)$ and $Z_\lambda(v)$ are convex polyhedra.*

Proof. Choose $v \in Z - \mathbb{R}^q_+$ and $x^0 \in X_\lambda(v)$. Then clearly,

$$X_\lambda(v) = \{x \in X | f(x) \geqslant v, \lambda^{\mathrm{T}} f(x) = \lambda^{\mathrm{T}} f(x^0)\}.$$

Moreover, Lemma 3.1 implies that $C_i x = C_i x^0$ holds for each $x \in X_\lambda(v)$, i.e., the vector $w^i(v) := C_i x^0$ is independent of x^0 and

$$f_i(x) = (w^i(v) + c^i)^{\mathrm{T}} x + \gamma_i, \quad x \in X_\lambda(v), i = 1, \dots, q.$$

Thus,

$$\begin{aligned} X_\lambda(v) = \{x \in X|\ & (w^i(v) + c^i)^{\mathrm{T}} x + \gamma_i \geqslant v_i,\ i = 1, \dots, q, \\ & \left(\sum_{i=1}^q \lambda_i (w^i(v) + c^i)\right)^{\mathrm{T}} x + \lambda^{\mathrm{T}} \gamma = \lambda^{\mathrm{T}} f(x^0), \\ & w^i(v) = C_i x,\ i = 1, \dots, q\}, \end{aligned} \tag{4}$$

i.e., $X_\lambda(v)$ is a convex polyhedron. Moreover, $Z_\lambda(v)$ is a convex polyhedron since f is linear on $X_\lambda(v)$. □

Now step (*ii*) of algorithm 4.1 is modified as follows:

(iia) Choose $N \in \mathbb{N}$ and compute a solution $x^0(i/N)$ to $\mathrm{MP}_\lambda(v(i/N))$, $i = 0, \dots, N$.

(iib) Compute the unique solution $z(i/N)$ to $\overline{\mathrm{MP}}_\lambda(v(i/N))$, $i = 0, \dots, N$. Set

$$D_i := (C_1 x^0(i/N) + c^1, \dots, C_q x^0(i/N) + c^q)^{\mathrm{T}} \in \mathbb{R}^{q \times n}, \quad i = 0, \dots, N, \tag{5}$$

and $\gamma := (\gamma_1, \dots, \gamma_q)^{\mathrm{T}} \in \mathbb{R}^q$, then for $x \in X_\lambda(v(i/N))$

$$\|f(x)\|^2 = \|D_i x + \gamma\|^2 = x^T D_i^T D_i x + 2\gamma^T D_i x + \gamma^T \gamma.$$

Consequently, $\overline{\mathrm{MP}}_\lambda(v(i/N))$ is equivalent to

$$\min!\{x^T D_i^T D_i x + 2\gamma^T D_i x | x \in X_\lambda(v(i/N))\}, \tag{6}$$

where $X_\lambda(v(i/N))$ is given by (4), i.e., it suffices to compute solutions $x(i/N)$ to this problem. Since $D_i^T D_i$, $i = 0, \ldots, N$, is positive semidefinite, these problems constitute convex quadratic optimization problems, which can be solved in polynomial time (cf. [16]). In summary, $z(i/N) := f(x(i/N))$, $i = 0, \ldots, N$, are points located on an efficient compromise arc.

6. The Domination Property of the Weakly Efficient Set

We conclude our paper by discussing the weak Domination Property, that is, the Domination Property for the set of weakly efficient points. Note that even for the convex MP, this property does in general not hold [17].

THEOREM 6.1. *Let $X \subset \mathbb{R}^n$ be a convex polyhedron, $f: \mathbb{R}^n \to \mathbb{R}^q$ be concave quadratic, $W(f(X))$ be nonempty. Then, the weak Domination Property holds.*

Proof. Choose $\lambda \in \mathbb{R}^q_+ \setminus \{0\}$ such that

$$\max!\{\lambda^T f(x) | x \in X\}$$

is solvable, i.e., the interval $I := \lambda^T f(X) - \mathbb{R}_+$ is bounded from above. Choose $v \in Z - \mathbb{R}^q_+$. Now the interval $J := \lambda^T f(X(v)) - \mathbb{R}_+$ is closed, since $\lambda^T f: \mathbb{R}^n \to \mathbb{R}$ is concave quadratic (cf. [12]). Moreover, we have that $J \subset I$, thus J is bounded from above. Consequently, there is a solution $x^1 \in X(v)$ to $\mathrm{MP}_\lambda(v)$, which is obviously weakly efficient in $X(v)$. □

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